CALCULUS II FINAL EXAM QUESTIONS

Student Number : ....

Name, Surname : 09.06.2016

1. Determine whether the series $\sum_{k=2}^{\infty} \frac{10}{k(\ln k)^2}$ converges or diverges.

Since $f(x)=\frac{10}{x(\ln x)^2}$ continuous, nonnegative and decreasing on the interval $[2,\infty)$, the integral test can be used.

$$\int_2^\infty \frac{10}{x(\ln x)^2}dx = \lim_{b\to\infty}\int_2^b \frac{10}{x(\ln x)^2}dx$$

$$\int \frac{10}{x(\ln x)^2}dx \quad \int \frac{10\,du}{u^2} = -\frac{10}{u} = -\frac{10}{\ln x} \qquad \ln x = u,\ \frac{dx}{x}=du$$

$$\Rightarrow \int_2^\infty \frac{10}{x(\ln x)^2}dx = \lim_{b\to\infty}\left(-\frac{10}{\ln x}\right)\Big|_2^b = \lim_{b\to\infty}\left[-\frac{10}{\ln b}^{\nearrow 0} + \frac{10}{\ln 2}\right] = \frac{10}{\ln 2}$$

Therefore the series is convergent.

2. Determine whether the given series converges:

$$\frac{1}{1\cdot 3}+\frac{1}{2\cdot 9}+\frac{1}{3\cdot 27}+\frac{1}{4\cdot 81}+\cdots$$

$$a_n = \frac{1}{n.3^n}$$

$$\frac{1}{n.3^n} < \frac{1}{3^n}$$

and

$$\sum_{n=1}^\infty \frac{1}{3^n} = \sum_{n=1}^\infty \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$$

Therefore the series is convergent.

3. Determine whether the series $\sum_{k=1}^{\infty} \frac{k!}{e^{k^2}}$ converges.

By ratio test,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)!}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{n!}$$

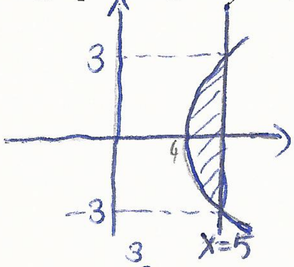
$$= \lim_{n\to\infty}\frac{(n+1).\cancel{n!}.e^{n^2}}{e^{n^2+2n+1}\,\cancel{n!}} = \lim_{n\to\infty}\frac{(n+1)\cancel{e^{n^2}}}{\cancel{e^{n^2}}.e^{2n}.e}$$

$$= \frac{1}{e}\lim_{n\to\infty}\frac{n+1}{e^{2n}} = \frac{1}{e}\lim_{n\to\infty}\frac{1}{2e^{2n}}$$

$$= 0 < 1$$

Therefore the series is convergent.

4. Find the volume of the solid of revolution that is formed by revolving the region bounded by the graphs of $x^2-y^2=16$ and $x=5$ about the y-axis.



$$x^2 = y^2+16 \Rightarrow x=\sqrt{y^2+16}$$
$$\Rightarrow 25 = y^2+16 \Rightarrow y=\mp 3$$

$$V = \pi\int_{-3}^{3}\left[5^2-\left(\sqrt{y^2+16}\right)^2\right]dy$$

$$= \pi\int_{-3}^{3}(25-y^2-16)\,dy = \pi\int_{-3}^{3}(9-y^2)\,dy$$

$$= \left[9y-\frac{y^3}{3}\right]\Big|_{-3}^{3}$$

$$= 36\pi \text{ br}^3$$

5. Find the area of an ellipse given by the formula $a^2x^2 + b^2y^2 = a^2b^2$.

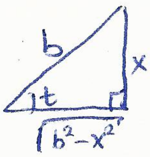
$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \Rightarrow y^2 = a^2 - \frac{a^2x^2}{b^2}$$

$$A = 4\int_0^b \sqrt{a^2 - \left(\frac{a}{b}x\right)^2}\,dx \qquad \frac{a}{b}x = a\sin t \Rightarrow dx = b\cos t\,dt$$

$$\int \sqrt{a^2-\left(\frac{a}{b}x\right)^2}\,dx = \int \sqrt{a^2 - a^2\sin^2 t}\; b\cos t\,dt$$

$$= \int ab\cos^2 t\,dt = ab\int \frac{1+\cos 2t}{2}\,dt$$

$$= \frac{ab}{2}\left[t + \frac{\sin 2t}{2}\right] = \frac{ab}{2}\left[t + \sin t\cos t\right]$$



$$= \frac{ab}{2}\left[\arcsin\frac{x}{b} + \frac{x\sqrt{b^2-x^2}}{b^2}\right]$$

$$A = 4\,\frac{ab}{2}\left[\arcsin\frac{x}{b} + \frac{x\sqrt{b^2-x^2}}{b^2}\right]\Bigg|_0^b$$

$$= 2ab\left[\arcsin(1) - \arcsin(0)\right]$$

$$= 2ab\left[\frac{\pi}{2} - 0\right] = \pi ab$$

6. Find the arc length of the graph of $y = \frac{1}{6}x^3 + \frac{1}{2x}$ on the interval $[2,4]$.

$$y' = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

$$(y')^2 = \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)$$

$$\Rightarrow L = \int_2^4 \sqrt{1+(y')^2}\,dx = \int_2^4 \sqrt{1 + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)}\,dx$$

$$= \int_2^4 \sqrt{\frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4x^4}}\,dx$$

$$= \int_2^4 \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2}\,dx$$

$$= \int_2^4 \left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)dx = \left[\frac{1}{2}\frac{x^3}{3} - \frac{1}{2x}\right]_2^4$$

$$= \frac{99}{8}\ \text{br}$$

7. Compare the exact value of integral $\int_0^1 \frac{1}{1+x^2}\,dx$ with the approximation obtained by Simpson's rule for $n = 4$.

$$n = 4 \Rightarrow \Delta x = \frac{1}{4}$$

$$x_0 = 0,\ x_1 = \frac{1}{4},\ x_2 = \frac{1}{2},\ x_3 = \frac{3}{4},\ x_4 =$$

$$S_4 = \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x$$

$$= \frac{1}{3.4}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1$$

$$= \frac{1}{12}\left[1 + 4.\frac{16}{17} + 2.\frac{4}{5} + 4.\frac{16}{25} + \frac{1}{2}\right]$$

$$= \frac{1}{12}\left[\frac{3}{2} + \frac{104}{25} + \frac{64}{17}\right] = \frac{8011}{10200}$$

$$\approx 0{,}785392$$

Exact value of the integral is

$$\int_0^1 \frac{dx}{1+x^2} = \arctan x\Big|_0^1$$

$$= \arctan(1) - \arctan(0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\approx 0{,}785398$$

Each right answer is 15 points. Duration of the exam is 80 minutes. Good luck.

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